10. cvičení - řešení

Příklad 1 (a)

$$\frac{2x+1}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3}$$
$$2x+1 = A(2x+3) + B(x+1) = 2Ax + 3A + Bx + B = (2A+B)x + 3A + B$$
$$\implies 2 = 2A + B$$
$$1 = 3A + B$$
$$B = 2 - 2A = 1 - 3A \implies A = -1, B = 4$$

$$\int \frac{2x+1}{(x+1)(2x+3)} \mathrm{d}x \overset{\text{lin.}}{=} -\int \frac{1}{x+1} \mathrm{d}x + 4 \int \frac{1}{2x+3} \mathrm{d}x \overset{\text{c}}{=} -\log|x+1| + 2\log|2x+3|$$

Příklad 1 (b)

$$\frac{x-1}{x(x+1)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$
$$x-1 = A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx$$
$$x-1 = A(x^3+3x^2+3x+1) + B(x^3+2x^2+x) + C(x^2+x) + Dx$$
$$x-1 = Ax^3 + 3Ax^2 + 3Ax + A + Bx^3 + 2Bx^2 + Bx + Cx^2 + Cx + Dx$$
$$x-1 = (A+B)x^3 + (3A+2B+C)x^2 + (3A+B+C+D)x + A$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$0 = A + B \overset{4}{\implies} B = 1$$
$$0 = 3A + 2B + C \overset{1+4}{\implies} C = 3 - 2 = 1$$
$$1 = 3A + B + C + D \overset{1+2+4}{\implies} D = 1 - 3A - B - C = 1 + 3 - 1 - 1 = 2$$
$$-1 = A$$

$$\int \frac{x-1}{x(x+1)^3} \mathrm{d}x \overset{\text{lin.}}{=} -\int \frac{1}{x} \mathrm{d}x + \int \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3} \mathrm{d}x =$$
$$= |y = x+1, \mathrm{d}y = \mathrm{d}x| = -\log|x| + \int \frac{1}{y} + \frac{1}{y^2} + \frac{2}{y^3} \mathrm{d}y =$$
$$\overset{\text{c}}{=} -\log|x| + \log|y| - \frac{1}{y} - \frac{1}{y^2} = -\log|x| + \log|x+1| - \frac{1}{(x+1)} - \frac{1}{(x+1)^2}$$

Příklad 1 (c)

$$\frac{x^{15}-3}{x-1} = x^{14}+x^{13}+\cdots+x+1-\frac{2}{x-1} \quad \text{dle dělení mnohočlenu mnohočlenem}$$

$$\int \frac{x^{15}-3}{x-1}\mathrm{d}x \overset{\text{lin.}}{=} \int x^{14}+x^{13}+\cdots+x+1\mathrm{d}x - \int \frac{2}{x-1}\mathrm{d}x \overset{\text{c}}{=} \frac{x^{15}}{15}+\frac{x^{14}}{14}+\cdots+\frac{x^2}{2}+x-2\log|x-1|$$

Příklad 1 (d)

$$\begin{aligned}
&\frac{x^4}{x^4+5x^2+4} = \frac{x^4+5x^2+4-5x^2-4}{x^4+5x^2+4} = 1-\frac{5x^2+4}{x^4+5x^2+4}\\
&x^4+5x^2+4 = (x^2+1)(x^2+4)
\end{aligned}$$

$$\begin{aligned}
&\frac{-5x^2-4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}\\
&-5x^2-4 = Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D\\
&-5x^2-4 = (A+C)x^3+(B+D)x^2+(4A+C)x+4B+D
\end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned}
&0 = A+C \overset{4}{\Longrightarrow} 0=-3A \Longrightarrow A=0, C=0\\
&-5 = B+D\\
&0 = 4A+C\\
&-4 = 4B+D \overset{2}{\Longrightarrow} -4 = 4B-5-B \Longrightarrow 3B=1 \Longrightarrow B=\frac{1}{3}, D=-\frac{16}{3}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{x^4}{x^4+5x^2+4}\mathrm{d}x \overset{\text{lin.}}{=} \int 1\mathrm{d}x + \frac{1}{3}\int \frac{1}{x^2+1}\mathrm{d}x - \frac{16}{3}\int \frac{1}{x^2+4}\mathrm{d}x =\\
&\overset{\text{lin.}}{=} x+\frac{1}{3}\arctan x - \frac{16}{3}\cdot\frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\mathrm{d}x = \left| y=\frac{x}{2}, \mathrm{d}y=\frac{1}{2}\mathrm{d}x \right| = x+\frac{1}{3}\arctan x - \frac{16}{6}\int \frac{1}{y^2+1}\mathrm{d}y =\\
&\overset{\text{c}}{=} x+\frac{1}{3}\arctan x - \frac{16}{6}\arctan y = x+\frac{1}{3}\arctan x - \frac{8}{3}\arctan\frac{x}{2}
\end{aligned}$$

Příklad 1 (e)

$$\begin{aligned}
&\frac{x^4}{x^4+2x^2-3} = \frac{x^4+2x^2-3}{x^4+2x^2-3}+\frac{-2x^2+3}{x^4+2x^2-3} = 1+\frac{-2x^2+3}{x^4+2x^2-3}\\
&x^4+2x^2-3 = (x^2-1)(x^2+3) = (x-1)(x+1)(x^2+3)
\end{aligned}$$

$$
\begin{aligned}
\frac{-2x^2+3}{(x-1)(x+1)(x^2+3)} &= \frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+3}\\
-2x^2+3 &= A(x-1)(x^2+3)+B(x+1)(x^2+3)+(Cx+D)(x^2-1)\\
-2x^2+3 &= Ax^3+3Ax-Ax^2-3A+Bx^3+3Bx+Bx^2+3B+Cx^3-Cx+Dx^2-D\\
-2x^2+3 &= (A+B+C)x^3+(-A+B+D)x^2+(3A+3B-C)x-3A+3B-D
\end{aligned}
$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$
\begin{aligned}
0 &= A+B+C\\
-2 &= -A+B+D\\
0 &= 3A+3B-C\\
3 &= -3A+3B-D
\end{aligned}
$$

První a třetí rovnice dávají: $0 = A+B+3A+3B = 4(A+B) \implies A+B=0 \stackrel{1}{\implies} C=0$. Speciálně $A=-B$. Dostáváme tedy násl.

$$
\begin{aligned}
C &= 0\\
-A &= B\\
-2 &= 2B+D\\
3 &= 6B-D
\end{aligned}
$$

Z posledních dvou rovnic plyne: $3-2 = 2B+6B = 8B \implies B=\frac{1}{8}$. Z třetí rovnice v poslední soustavě pak dostáváme: $D=-2-2B=-2-\frac{1}{4}=-\frac{9}{4}$.

Máme tedy: $A=-\frac{1}{8}, B=\frac{1}{8}, C=0, D=-\frac{9}{4}$

$$
\begin{aligned}
&\int \frac{x^4}{x^4+2x^2-3}\mathrm{d}x \stackrel{\text{lin.}}{=} \int 1\mathrm{d}x - \frac{1}{8}\int\frac{1}{x+1}\mathrm{d}x+\frac{1}{8}\int\frac{1}{x-1}\mathrm{d}x-\frac{9}{4}\int\frac{1}{x^2+3}\mathrm{d}x=\\
&=|y=x+1,\mathrm{d}y=\mathrm{d}x, z=x-1,\mathrm{d}z=\mathrm{d}x| = x-\frac{1}{8}\int\frac{1}{y}\mathrm{d}y+\frac{1}{8}\int\frac{1}{z}\mathrm{d}z-\frac{9}{4}\cdot\frac{1}{3}\int\frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\mathrm{d}x=\\
&=\left|u=\frac{x}{\sqrt{3}},\mathrm{d}u=\frac{1}{\sqrt{3}}\mathrm{d}x\right| \stackrel{\text{lin.}}{=} x-\frac{1}{8}\log|y|+\frac{1}{8}\log|z|-\frac{3\sqrt{3}}{4}\int\frac{1}{1+u^2}\mathrm{d}u=\\
&\stackrel{\text{c}}{=} x-\frac{1}{8}\log|y|+\frac{1}{8}\log|z|-\frac{3\sqrt{3}}{4}\arctan u = x-\frac{1}{8}\log|x+1|+\frac{1}{8}\log|x-1|-\frac{3\sqrt{3}}{4}\arctan\frac{x}{\sqrt{3}}
\end{aligned}
$$

Příklad 1 (f)

$$
\begin{aligned}
&\frac{1}{(x^2-4x+4)(x^2-4x+5)} = \frac{1}{(x-2)^2(x^2+4x+5)}\\
&\frac{1}{(x-2)^2(x^2+4x+5)} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2-4x+5}\\
&1 = A(x-2)(x^2-4x+5)+B(x^2-4x+5)+(Cx+D)(x^2-4x+4)\\
&1 = Ax^3-4Ax^2+5Ax-2Ax^2+8Ax-10A+Bx^2-4Bx+5B+Cx^3-4Cx^2+4Cx+Dx^2-4Dx+4D\\
&1 = (A+C)x^3+(-4A-2A+B-4C+D)x^2+(5A+8A-4B+4C-4D)x-10A+5B+4D
\end{aligned}
$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned}
0 &= A + C \implies C = -A \\
0 &= -6A + B - 4C + D \overset{1}{\implies} 0 = 2C + B + D \\
0 &= 13A - 4B + 4C - 4D \overset{1}{\implies} 0 = -9C - 4B - 4D \\
1 &= -10A + 5B + 4D \overset{1}{\implies} 0 = 10C + 5B + 4D
\end{aligned}$$

Poslední dvě rovnice dávají: $1 = -9C - 4B + 10C + 5B = C + B \implies C = -B + 1$. Ze druhé rovnice pak plyne: $0 = -2B + 2 + B + D = 2 - B + D \implies D = B - 2$. Ze třetí rovnice pak plyne: $0 = 9B - 9 - 4B - 4B + 8 = B - 1 \implies B = 1, D = -1, C = 0, A = 0$.

$$\begin{aligned}
&\int \frac{1}{(x^2 - 4x + 4)(x^2 - 4x + 5)} \mathrm{d}x \overset{\text{lin.}}{=} \int \frac{1}{(x-2)^2} \mathrm{d}x - \int \frac{1}{x^2 - 4x + 5} \mathrm{d}x = \\
&= \int \frac{1}{(x-2)^2} \mathrm{d}y - \int \frac{1}{(x-2)^2 + 1} \mathrm{d}x = |y = x - 2, \mathrm{d}y = \mathrm{d}x| = \int \frac{1}{y^2} \mathrm{d}y - \int \frac{1}{y^2 + 1} \mathrm{d}y = \\
&\overset{\text{c}}{=} \frac{-1}{y} - \arctan y = -\frac{1}{x-2} - \arctan(x - 2)
\end{aligned}$$

Příklad 1 (g)

$$\begin{aligned}
&\frac{x^2}{(x^2 + 2x + 2)^2} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{(x^2 + 2x + 2)^2} \\
&x^2 = (Ax + B)(x^2 + 2x + 2) + Cx + D \\
&x^2 = Ax^3 + 2Ax^2 + 2Ax + Bx^2 + 2Bx + 2B + Cx + D \\
&x^2 = Ax^3 + (2A + B)x^2 + (2A + 2B + C)x + 2B + D
\end{aligned}$$

Porovnáním koeficientů u jednotlivých mocnin x dostáváme soustavu:

$$\begin{aligned}
0 &= A \\
1 &= 2A + B \overset{1}{\implies} B = 1 \\
0 &= 2A + 2B + C \overset{1+2}{\implies} C = -2 \\
0 &= 2B + D \overset{2}{\implies} D = -2
\end{aligned}$$

$$\begin{aligned}
&\int \frac{x^2}{(x^2 + 2x + 2)^2} \mathrm{d}x \overset{\text{lin.}}{=} \int \frac{1}{x^2 + 2x + 2} \mathrm{d}x - 2 \int \frac{x + 1}{(x^2 + 2x + 2)^2} \mathrm{d}x \overset{\text{lin.}}{=} \int \frac{1}{(x+1)^2 + 1} \mathrm{d}x - \int \frac{2x + 2}{(x^2 + 2x + 2)^2} \mathrm{d}x = \\
&= \left|y = x + 1, \mathrm{d}y = \mathrm{d}x, z = x^2 + 2x + 2, \mathrm{d}z = (2x + 2)\mathrm{d}x\right| = \int \frac{1}{y^2 + 1} \mathrm{d}y - \int \frac{1}{z^2} \mathrm{d}z = \arctan y + \frac{1}{z} = \\
&= \arctan(x + 1) + \frac{1}{x^2 + 2x + 2}
\end{aligned}$$

Níže je uvedeno jen částečné řešení

rozklad na parciální zlomky je algoritmický a snadno jej provedete sami

Příklad 1 (h)

$$\int \frac{x+1}{x^2-x+1}\mathrm{d}x = \left|y = x^2-x+1, \mathrm{d}y = (2x-1)\,\mathrm{d}x\right| = \int \frac{2(x+1)}{2(x^2-x+1)}\mathrm{d}x = \int \frac{2x+2-3+3}{2(x^2-x+1)}\mathrm{d}x =$$

$$\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\mathrm{d}x + \frac{3}{2}\int \frac{1}{x^2-x+1}\mathrm{d}x = \frac{1}{2}\int \frac{1}{y}\mathrm{d}y + \frac{3}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x =$$

$$= \frac{1}{2}\log|y| + \frac{3}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}x = \left|z = \sqrt{\frac{4}{3}}\left(x-\frac{1}{2}\right), \mathrm{d}z = \sqrt{\frac{4}{3}}\right| =$$

$$= \frac{1}{2}\log|x^2-x+1| + 2\sqrt{\frac{3}{4}}\int \frac{1}{z^2+1}\mathrm{d}z \overset{\text{c}}{=} \frac{1}{2}\log|x^2-x+1| + \sqrt{3}\arctan z =$$

$$= \frac{1}{2}\log|x^2-x+1| + \sqrt{3}\arctan\left(\sqrt{\frac{4}{3}}\left(x-\frac{1}{2}\right)\right) = \frac{1}{2}\log|x^2-x+1| + \sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\cdot\frac{2x-1}{2}\right)$$

$$\int \frac{1}{x+1}\mathrm{d}x = |y = x+1, \mathrm{d}y = \mathrm{d}x| = \int \frac{1}{y}\mathrm{d}y \overset{\text{c}}{=} \log|y| = \log|x+1|$$

Příklad 1 (i)

$$\int \frac{-3x+2}{x^2-x+2}\,\mathrm{d}x = \left|y = x^2-x+2, \mathrm{d}y = 2x-1\mathrm{d}x\right| = \int \frac{-\frac{2}{3}}{-\frac{2}{3}}\cdot\frac{-3x+2}{x^2-x+2}\mathrm{d}x =$$

$$= -\frac{3}{2}\int \frac{2x-\frac{4}{3}+1-1}{x^2-x+2}\mathrm{d}x \overset{\text{lin.}}{=} -\frac{3}{2}\int \frac{2x-1}{x^2-x+2}\mathrm{d}x - \frac{3}{2}\cdot\frac{-1}{3}\int \frac{1}{x^2-x+2}\mathrm{d}x =$$

$$= -\frac{3}{2}\int \frac{1}{y}\mathrm{d}y + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\mathrm{d}x = -\frac{3}{2}\log|y| + \frac{1}{2}\cdot\frac{4}{7}\int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)^2+1}\mathrm{d}x =$$

$$= \left|z = \sqrt{\frac{4}{7}}\left(x-\frac{1}{2}\right), \mathrm{d}z = \sqrt{\frac{4}{7}}\mathrm{d}x\right| = -\frac{3}{2}\log|x^2-x+2| + \frac{2}{7}\cdot\sqrt{\frac{7}{4}}\int \frac{1}{z^2+1}\mathrm{d}z =$$

$$\overset{\text{c}}{=} -\frac{3}{2}\log|x^2-x+2| + \frac{\sqrt{7}}{7}\arctan z = -\frac{3}{2}\log|x^2-x+2| + \frac{1}{\sqrt{7}}\arctan\left(\sqrt{\frac{4}{7}}\left(x-\frac{1}{2}\right)\right) =$$

$$= -\frac{3}{2}\log|x^2-x+2| + \frac{1}{\sqrt{7}}\arctan\left(\frac{2}{\sqrt{7}}\cdot\frac{2x-1}{2}\right) = -\frac{3}{2}\log|x^2-x+2| + \frac{1}{\sqrt{7}}\arctan\frac{2x-1}{\sqrt{7}}$$

Příklad 1 (j)

$$\int \frac{x+1}{x^2+x+1}\mathrm{d}x = \left|y = x^2+x+1, \mathrm{d}y = (2x+1)\mathrm{d}x\right| = \int \frac{2}{2}\cdot\frac{x+1}{x^2+x+1}\mathrm{d}x \overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x+2-1+1}{x^2+x+1}\mathrm{d}x =$$
$$\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\mathrm{d}x + \frac{1}{2}\int \frac{1}{x^2+x+1}\mathrm{d}x = \frac{1}{2}\int \frac{1}{y}\mathrm{d}y + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x =$$
$$= \frac{1}{2}\log|y| + \frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}x = \left|z = \sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right), \mathrm{d}z = \sqrt{\frac{4}{3}}\mathrm{d}x\right| =$$
$$= \frac{1}{2}\log|x^2+x+1| + \frac{2}{3}\sqrt{\frac{3}{4}}\int \frac{1}{z^2+1}\mathrm{d}z \overset{\text{c}}{=} \frac{1}{2}\log|x^2+x+1| + \frac{\sqrt{3}}{3}\arctan z =$$
$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right) = \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\frac{2x+1}{2} =$$
$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$\int \frac{x-1}{x^2-x+1}\mathrm{d}x = \left|y = x^2-x+1, \mathrm{d}y = (2x-1)\mathrm{d}x\right| \overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x-2+1-1}{x^2-x+1}\mathrm{d}x =$$
$$\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\mathrm{d}x - \frac{1}{2}\int \frac{1}{x^2-x+1}\mathrm{d}x = \frac{1}{2}\int \frac{1}{y}\mathrm{d}x - \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x =$$
$$= \frac{1}{2}\log|y| - \frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}x = \left|z = \sqrt{\frac{4}{3}}\left(x-\frac{1}{2}\right), \mathrm{d}z = \sqrt{\frac{4}{3}}\mathrm{d}x\right| =$$
$$= \frac{1}{2}\log|x^2-x+1| - \frac{2}{3}\sqrt{\frac{3}{4}}\int \frac{1}{z^2+1}\mathrm{d}z \overset{\text{c}}{=} \frac{1}{2}\log|x^2-x+1| - \frac{1}{\sqrt{3}}\arctan z =$$
$$= \frac{1}{2}\log|x^2-x+1| - \frac{1}{\sqrt{3}}\arctan\sqrt{\frac{4}{3}}\left(x-\frac{1}{2}\right) = \frac{1}{2}\log|x^2-x+1| - \frac{1}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\frac{2x-1}{2} =$$
$$= \frac{1}{2}\log|x^2-x+1| - \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$

Příklad 1 (k)

$$\int \frac{1}{2x+1}\mathrm{d}x = |y = 2x+1, \mathrm{d}y = 2\mathrm{d}x| = \int \frac{1}{2y}\mathrm{d}y \overset{\text{c}}{=} \frac{1}{2}\log|y| = \frac{1}{2}\log|2x+1|$$

$$\int \frac{2x+1}{x^2+x+1}\mathrm{d}x = \left|y = x^2+x+1, \mathrm{d}y = (2x+1)\mathrm{d}x\right| = \int \frac{1}{y}\mathrm{d}y \overset{\text{c}}{=} \log|y| = \log|x^2+x+1|$$

$$\int \frac{7x+5}{\left(x^2+x+1\right)^2}\mathrm{d}x = \left|y = x^2+x+1, \mathrm{d}y = (2x+1)\mathrm{d}x\right| \overset{\text{lin.}}{=} 7\int \frac{x+\frac{5}{7}}{\left(x^2+x+1\right)^2}\mathrm{d}x =$$
$$= \frac{7}{2}\int \frac{2x+\frac{10}{7}-\frac{3}{7}+\frac{3}{7}}{\left(x^2+x+1\right)^2}\mathrm{d}x \overset{\text{lin.}}{=} \frac{7}{2}\int \frac{2x+1}{\left(x^2+x+1\right)^2}\mathrm{d}x + \frac{7}{2}\cdot\frac{3}{7}\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x =$$
$$= \frac{7}{2}\int \frac{1}{y^2}\mathrm{d}y + \frac{3}{2}\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x = -\frac{7}{2y} + \frac{3}{2}\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x = -\frac{7}{2(x^2+x+1)} + \frac{3}{2}\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x$$

$$\int \frac{1}{(x^2+x+1)^2}\mathrm{d}x = \int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\mathrm{d}x = \frac{16}{9}\int \frac{1}{\left(\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1\right)^2}\mathrm{d}x =$$

$$= \left| u = \sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right), \mathrm{d}u = \sqrt{\frac{4}{3}}\mathrm{d}x \right| = \frac{16}{9}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{(u^2+1)^2}\mathrm{d}u = \left| u = \tan t, \mathrm{d}u = \frac{1}{\cos^2 t}\mathrm{d}t \right| =$$

$$= \frac{8\sqrt{3}}{9}\int \frac{1}{(\tan^2+1)^2}\frac{1}{\cos^2 t}\mathrm{d}t = \frac{8\sqrt{3}}{9}\int \frac{1}{\left(\frac{\sin^2 t+\cos^2 t}{\cos^2 t}\right)^2}\frac{1}{\cos^2 t}\mathrm{d}t = \frac{8\sqrt{3}}{9}\int \cos^2 t\mathrm{d}t = \frac{8\sqrt{3}}{9}\frac{1}{2}(\cos t\sin t + t)$$

Výše se využilo následujícího:

$$\int \cos^2 x\mathrm{d}x = \int \cos x\cdot\cos x\mathrm{d}x \overset{\text{PP}}{=} \sin x\cos x - \int \sin x(-\sin x)\mathrm{d}x = \sin x\cos x + \int \sin^2 x\mathrm{d}x =$$
$$= \sin x\cos x + \int 1-\cos^2 x\mathrm{d}x \overset{\text{lin.}}{=} \sin x\cos x + x - \int \cos^2 x\mathrm{d}x$$

Vyjádřením ze získané rovnosti dostáváme:

$$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\left(\sin x\cos x + x\right)$$

Přičemž: $u = \tan t$, tedy $t = \arctan t$. Navíc $1+\tan^2 t = \frac{1}{\cos^2 t}$, tedy $\cos^2 t = \frac{1}{\tan^2 t+1}$. Z toho plyne, že $\sin t\cos t = \frac{\sin t}{\cos t}\cos^2 t = \tan t\frac{1}{\tan^2 t+1}$, tedy $\cos t\sin t = \frac{u}{u^2+1}$. Z toho plyne, že:

$$\int \frac{1}{(x^2+x+1)^2}\mathrm{d}x \overset{\text{c}}{=} \frac{4\sqrt{3}}{9}\left(\arctan u + \frac{u}{u^2+1}\right) = \frac{4\sqrt{3}}{9}\left(\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)}{\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right)^2+1}\right)$$

Příklad 1 (l)
Rovnice $x^4+x^2+1=0$ je reciproká. Jak na reciproké rovnice vizte zde:
https://www2.karlin.mff.cuni.cz/~ portal/komplexni_cisla.dp/?page=rovnice-reciproke

$$\begin{aligned}
&x^4+x^2+1=0 \quad \text{vydělíme výrazem } x^2\\
&x^2+1+\frac{1}{x^2}=0, \quad y := x+\frac{1}{x}, y^2 = \left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}\\
&\implies y^2-1=0\\
&(y+1)(y-1)=0
\end{aligned}$$

Rovnice $x^4+x^2+1=0$ lze substitucí $y = x+\frac{1}{x}$ přepsat na $(y+1)(y-1)$. Po provedení desubstituce dostáváme rozklad: $x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$.

Rozklad na parciální zlomky získáme vyšetřením rovnosti:

$$\frac{x^2+1}{(x^4+x^2+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{x^2-x+1} + \frac{Gx+H}{(x^2-x+1)^2}$$

Dostaneme: $A = B = D = E = F = H = \frac{1}{4}, C = G = -\frac{1}{4}$.

$$\int \frac{x+1}{x^2+x+1}\mathrm{d}x = \left|y = x^2+x+1, \mathrm{d}y = (2x+1)\mathrm{d}x\right| = \frac{1}{2}\int \frac{2x+2-1+1}{x^2+x+1}\mathrm{d}x =$$
$$\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\mathrm{d}x + \frac{1}{2}\int \frac{1}{x^2+x+1}\mathrm{d}x = \frac{1}{2}\int \frac{1}{y}\mathrm{d}y + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x =$$
$$\frac{1}{2}\log|y| + \frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}x = \left|z = \sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right), \mathrm{d}z = \sqrt{\frac{4}{3}}\mathrm{d}x\right| =$$
$$= \frac{1}{2}\log|x^2+x+1| + \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{z^2+1}\mathrm{d}z \overset{\text{c}}{=} \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan z =$$
$$\frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$\int \frac{x+1}{(x^2+x+1)^2}\mathrm{d}x = \left|y = x^2+x+1, \mathrm{d}y = (2x+1)\mathrm{d}x\right| = \frac{1}{2}\int \frac{2x+2-1+1}{(x^2+x+1)^2}\mathrm{d}x =$$
$$\overset{\text{lin.}}{=} \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)^2}\mathrm{d}x + \frac{1}{2}\int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{y^2}\mathrm{d}y + \frac{1}{2}\int \frac{1}{\frac{9}{16}\left(1+\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2\right)^2}\mathrm{d}x =$$
$$= \left|u = \sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right), \mathrm{d}u = \sqrt{\frac{4}{3}}\mathrm{d}x\right| = -\frac{1}{2y} + \frac{1}{2}\cdot\frac{16}{9}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{(u^2+1)^2}\mathrm{d}u =$$
$$= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\int \frac{1}{(u^2+1)^2}\mathrm{d}u = \left|u = \tan t, \mathrm{d}u = \frac{1}{\cos^2 t}\mathrm{d}t\right| =$$
$$= -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\int \frac{1}{(\tan^2 t+1)^2\cos^2 t}\mathrm{d}t = -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\int \frac{1}{\left(\frac{\sin^2+\cos^2}{\cos^2}\right)^2\cos^2 t}\mathrm{d}t =$$
$$-\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\int \cos^2 t\mathrm{d}t$$

Využijeme následujícího:

$$\int \cos^2 x\mathrm{d}x = \int \cos x\cdot\cos x\mathrm{d}x \overset{\text{PP}}{=} \sin x\cos x - \int \sin x(-\sin x)\mathrm{d}x = \sin x\cos x + \int \sin^2 x\mathrm{d}x =$$
$$= \sin x\cos x + \int 1-\cos^2 x\mathrm{d}x \overset{\text{lin.}}{=} \sin x\cos x + x - \int \cos^2 x\mathrm{d}x$$

Vyjádřením ze získané rovnosti dostáváme:

$$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\left(\sin x\cos x + x\right)$$

Pokračujeme ve výpočtu:

$$\int \frac{x+1}{(x^2+x+1)^2}\mathrm{d}x = -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\int \cos^2 t \mathrm{d}t \overset{\mathrm{c}}{=} -\frac{1}{2(x^2+x+1)} + \frac{4\sqrt{3}}{9}\frac{1}{2}\left(\sin t \cos t + t\right) =$$

$$= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9}\left(\tan t \cos^2 t + t\right) =$$

$$= \left| 1 + \tan^2 t = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \implies \cos^2 t = \frac{1}{\tan^2 t + 1} \right| =$$

$$= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9}\left(\frac{u}{u^2+1} + \arctan u\right) =$$

$$= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9}\left(\frac{\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \arctan \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) =$$

$$= -\frac{1}{2(x^2+x+1)} + \frac{2\sqrt{3}}{9}\left(\frac{\frac{2x+1}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \arctan \frac{2x+1}{\sqrt{3}}\right) =$$

Zbytek analogicky.